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Fuzzy Mass Relations for the Higgs

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Abstract

The non-commutative approach of the standard model produces a relation between the top and the Higgs masses. We show that, for a given top mass, the Higgs mass is constrained to lie in an interval. The length of this interval is of the order of m_τ^2/m_t.

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We believe that non-commutative geometry [1] is about to revolutionize physics [1], [2] to the same extent as Minkowskian and Riemannian geometry did. A first clear prediction of the new theory is the value of the Higgs mass [3], $m_H = 280 \pm 33 \ GeV$, if the top mass is $m_t = 176 \pm 18 \ GeV$. The aim of this review is to appreciate the concepts behind this numerical constraint.

Let us view a Yang-Mills-Higgs model as a point in an infinite discrete space and a real parameter space. The points are labeled by an arbitrary finite dimensional, real, compact Lie group G, three arbitrary unitary representations ρ_L, ρ_R, ρ_S and several multi-linear invariants of order two, three and four. The group describes the gauge interactions. The representations describe the spectrum of left- and right-handed fermions and of scalars. The invariants are parameterized by gauge couplings, Yukawa couplings and scalar self-couplings and the parameter space is some Cartesian power of the real line. This power depends on the point in the discrete space. Today's dilemma of particle physics can be summarized as follows: Experiment has singled out a mediocre point in the discrete space, the standard model. Its real parameter space is 18 dimensional and without any structure. Namely, the standard model contains 3 gauge couplings, masses for the W, 3 leptons, 6 quarks, 1 scalar and 4 Cabbibo-Kobayashi-Maskawa mixing parameters, that all remain arbitrary.

In the non-commutative Connes-Lott approach to Yang-Mills-Higgs models, the entire Higgs sector comes free of charge. Thereby both the discrete and the real parameter space is reduced tremendously [4]. While the two Yang-Mills-Higgs spaces are hypercubes, both Connes-Lott spaces have a rich structure. In particular, the Higgs mass is forces into an interval. The length is of the order of m_τ^2 / m_t.

1 Yang-Mills-Higgs models

Let us first set up our notations of a YMH model. It is defined by the following input:

- a finite dimensional, real, compact Lie group G,

- an invariant scalar product on the Lie algebra $\mathfrak{g}$ of G, this choice being parameterized by a few positive numbers g_i, the gauge coupling,

- a (unitary) representation ρ_L on a Hilbert space $\mathcal{H}_L$ accommodating the left handed fermions ψ_L,

- a representation ρ_R on $\mathcal{H}_R$ for the right handed fermions ψ_R,

- a representation ρ_S on $\mathcal{H}_S$ for the scalars φ,

- an invariant, positive polynomial $V(\varphi)$, $\varphi \in \mathcal{H}_S$ of order 4, the Higgs potential,

- one complex number or Yukawa coupling g_Y for every trilinear invariant, i.e. for every one dimensional invariant subspace, "singlet", in the decomposition of the representation associated to $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S) \oplus (\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S^*)$.

The standard model is defined by the following input:

$$G = SU(3) \times SU(2) \times U(1)$$

with three coupling constants g_3, g_2, g_1 defined conventionally by

$$(b_1, b_2) := \frac{1}{g_1^2} \bar{b}_1 b_2, \quad b_1, b_2 \in u(1),$$

$$(a_1, a_2) := \frac{2}{g_n^2} \operatorname{tr}(a_2^* a_2), \quad a_1, a_2 \in su(n).$$

The representations are

$$\mathcal{H}_L = \bigoplus_1^3 \left[(1, 2, -1) \oplus (3, 2, \tfrac{1}{3}) \right],$$

$$\mathcal{H}_R = \bigoplus_1^3 \left[(1, 1, -2) \oplus (3, 1, \tfrac{4}{3}) \oplus (3, 1, -\tfrac{2}{3}) \right],$$

$$\mathcal{H}_S = (1, 2, -1/2),$$

where (n_3, n_2, y) denotes the tensor product of an n_3 dimensional representation of $SU(3)$, an n_2 dimensional representation of $SU(2)$ and the one dimensional representation of $U(1)$ with hypercharge y:

$$\rho(e^{i\theta}) = e^{iy\theta}, \qquad y \in \mathbb{Q}, \ \theta \in [0, 2\pi),$$
$$V(\varphi) = \lambda(\varphi^* \varphi)^2 - \frac{\mu^2}{2} \varphi^* \varphi, \qquad \varphi \in \mathcal{H}_S, \quad \lambda, \mu > 0.$$

There are 27 Yukawa couplings in the standard model, that can be traded in for 9 fermion masses and 4 mixing parameters.

The gauge symmetry is said to be spontaneously broken if every minimum $v \in \mathcal{H}_S$ of the Higgs potential is gauge variant, $\rho_S(g)v \neq v$ for some $g \in G$. Any such minimum v is called a vacuum. For simplicity let us assume that the vacuum is non-degenerate, i.e. all minima lie on the same orbit under G. To do perturbation theory, we have to introduce a scalar variable h, that vanishes in the vacuum,

$$h(x) := \varphi(x) - v,$$

x a point in spacetime M. With this change of variables, the Klein-Gordon Lagrangian is $(\mathrm{D}\varphi)^* * \mathrm{D}\varphi$. The Hodge star $*\cdot$ should be distinguished from the Hilbert space dual $\cdot^*$, wedge symbols are suppressed. We denote by D the covariant exterior derivative, here for scalars

$\mathrm{D}\varphi := \mathrm{d}\varphi + \tilde{\rho}_S(A)\varphi$, φ is now a multiplet of *fields*, i.e. a 0-form on spacetime with values in the scalar representation space, $\varphi \in \Omega^0(M, \mathcal{H}_S)$, while the vacuum v remains constant over spacetime so that it also minimizes the kinetic term $\mathrm{d}\varphi^* * \mathrm{d}\varphi$. The gauge fields are 1-forms with values in the Lie algebra of G: $A \in \Omega^1(M, \mathfrak{g})$, $\tilde{\rho}_S$ denotes the Lie algebra representation on $\mathcal{H}_S$. The Klein-Gordon Lagrangian produces the mass matrix for the gauge bosons A. This mass matrix is given by the (constant) symmetric, positive semi-definite form on the Lie algebra of G,

$$(\tilde{\rho}_S(A)v)^* \, \tilde{\rho}_S(A)v.$$

It contains the masses of the gauge bosons some of which remain massless. In the example of the standard model, these are the photon and the gluons.

In the following we are more concerned with the fermionic mass matrix $\mathcal{M}$, a linear map $\mathcal{M} : \mathcal{H}_R \longrightarrow \mathcal{H}_L$. We want to produce it in the same way we produced the mass matrix for the gauge bosons, via the change of variables $h(x) := \varphi(x) - v$. For this purpose, we add by hand to the Dirac Lagrangian gauge invariant trilinears

$$\sum_{j=1}^{n} g_{Yj} \left(\psi_L^*, \psi_R, \varphi \right)_j + \sum_{j=n+1}^{m} g_{Yj} \left(\psi_L^*, \psi_R, \varphi^* \right)_j \; + \; \text{complex conjugate}, \tag{1}$$

n is the number of singlets in $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S)$, $m+n$ the number of singlets in $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S^*)$. For $h = 0$ again, we obtain the fermionic mass matrix $\mathcal{M}$ as a function of the Yukawa couplings g_{Yj} and the vacuum v

$$\psi_L^* \mathcal{M} \psi_R := \sum_{j=1}^{n} g_{Yj} \left(\psi_L^*, \psi_R, v \right)_j + \sum_{j=n+1}^{m} g_{Yj} \left(\psi_L^*, \psi_R, v^* \right)_j.$$

As the gauge boson masses, the fermionic mass terms $\psi_L^* \mathcal{M} \psi_R$ are not gauge invariant in general. They are gauge invariant if $\rho_L(g^{-1})\mathcal{M}\rho_R(g) = \mathcal{M}$ for all $g \in G$. In the standard model with its 27 Yukawa couplings, the mass matrix $\mathcal{M}$ can be any matrix yielding mass terms invariant under $SU(3) \times U(1)$.

2 Connes-Lott models

This section summarizes the non-commutative approach to Yang-Mills-Higgs models, [1], [2], [5], [6]. Although we shall follow this approach due to Connes and Lott, let us mention that there are alternative approaches similar in spirit, [7], [8], [9].

2.1 Internal space

A Connes-Lott model is defined by the following choices:

- a finite dimensional, associative, algebra $\mathcal{A}$ over the field $\mathbb{R}$ or $\mathbb{C}$ with unit 1 and involution *,

- two *- representations of $\mathcal{A}$, ρ_L and ρ_R, on Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$ over the field, such that $\rho := \rho_L \oplus \rho_R$ is faithful,

- a mass matrix $\mathcal{M}$ i.e. a linear map $\mathcal{M} : \mathcal{H}_R \longrightarrow \mathcal{H}_L$,

- a certain number of coupling constants depending on the degree of reducibility of $\rho_L \oplus \rho_R$.

The data $(\mathcal{H}_L, \mathcal{H}_R, \mathcal{M})$ plays a fundamental role in non-commutative Riemannian geometry where it is called K-cycle.

With this input and the rules of non-commutatative geometry, Connes and Lott construct a YMH model. Their starting point is an auxiliary differential algebra $\Omega\mathcal{A}$, the so called universal differential envelope of $\mathcal{A}$:

$$\Omega^0 \mathcal{A} := \mathcal{A},$$

$\Omega^1 \mathcal{A}$ is generated by symbols δa, $a \in \mathcal{A}$ with relations

$$\delta 1 = 0, \quad \delta(ab) = (\delta a)b + a\delta b.$$

Therefore $\Omega^1 \mathcal{A}$ consists of finite sums of terms of the form $a_0 \delta a_1$,

$$\Omega^1 \mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j, \quad a_0^j, a_1^j \in \mathcal{A} \right\}$$

and likewise for higher p,

$$\Omega^p \mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j ... \delta a_p^j, \quad a_q^j \in \mathcal{A} \right\}.$$

The differential δ is defined by $\delta(a_0 \delta a_1 ... \delta a_p) := \delta a_0 \delta a_1 ... \delta a_p$.

Two remarks: The universal differential envelope $\Omega\mathcal{A}$ of a commutative algebra $\mathcal{A}$ is not necessarily graded commutative. The universal differential envelope of any algebra has no cohomology. This means that every closed form ω of degree $p \geq 1$, $\quad \delta\omega = 0$, is exact, $\omega = \delta\kappa$ for some $(p-1)$form κ.

The involution * is extended from the algebra $\mathcal{A}$ to $\Omega^1 \mathcal{A}$ by putting

$$(\delta a)^* := \delta(a^*) =: \delta a^*.$$

With the definition $(\omega\kappa)^* = \kappa^* \omega^*$, the involution is extended to the whole differential envelope.

The next step is to extend the representation $\rho := \rho_L \oplus \rho_R$ on $\mathcal{H} := \mathcal{H}_L \oplus \mathcal{H}_R$ from the algebra $\mathcal{A}$ to its universal differential envelope $\Omega\mathcal{A}$. This extension is the central piece of Connes' algorithm and deserves a new name:

$$\pi : \Omega\mathcal{A} \longrightarrow \text{End}(\mathcal{H})$$

$$\pi(a_0 \delta a_1 ... \delta a_p) := (-i)^p \rho(a_0)[\mathcal{D}, \rho(a_1)]...[\mathcal{D}, \rho(a_p)]$$

where $\mathcal{D}$ is the linear map from $\mathcal{H}$ into itself

$$\mathcal{D} := \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^* & 0 \end{pmatrix}.$$

In non-commutative geometry, $\mathcal{D}$ plays the role of the Dirac operator and we call it internal Dirac operator. A straightforward calculation shows that π is in fact a representation of $\Omega\mathcal{A}$ as involution algebra, and we are tempted to define also a differential, again denoted by δ, on $\pi(\Omega\mathcal{A})$ by

$$\delta\pi(\omega) := \pi(\delta\omega). \tag{2}$$

However, this definition does not make sense if there are forms $\omega \in \Omega\mathcal{A}$ with $\pi(\omega) = 0$ and $\pi(\delta\omega) \neq 0$. By dividing out these unpleasant forms, Connes constructs a new differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$, the interesting object:

$$\Omega_{\mathcal{D}}\mathcal{A} := \frac{\pi(\Omega\mathcal{A})}{J}$$

with

$$J := \pi(\delta \ker \pi) =: \bigoplus_p J^p$$

(J for junk). On the quotient now, the differential (2) is well defined. Degree by degree we have:

$$\Omega_{\mathcal{D}}^0\mathcal{A} = \rho(\mathcal{A})$$

because $J^0 = 0$,

$$\Omega_{\mathcal{D}}^1\mathcal{A} = \pi(\Omega^1\mathcal{A})$$

because ρ is faithful, and in degree $p \geq 2$,

$$\Omega_{\mathcal{D}}^p\mathcal{A} = \frac{\pi(\Omega^p\mathcal{A})}{\pi(\delta(\ker \pi)^{p-1})}.$$

While $\Omega\mathcal{A}$ has no cohomology, $\Omega_{\mathcal{D}}\mathcal{A}$ does in general. In fact, in infinite dimensions, if $\mathcal{F}$ is the algebra of complex functions on spacetime M and if the K-cycle is obtained from the Dirac operator then $\Omega_{\partial\!\!\!/}\mathcal{F}$ is de Rham's differential algebra of differential forms on M.

We come back to our finite dimensional case. Remember that the elements of the auxiliary differential algebra $\Omega\mathcal{A}$ that we introduced for book keeping purposes only, are abstract entities defined in terms of symbols and relations. On the other hand the elements of $\Omega_{\mathcal{D}}\mathcal{A}$, the "forms", are operators on the Hilbert space $\mathcal{H}$, i.e. concrete matrices of complex numbers. Therefore there is a natural scalar product defined by

$$< \hat{\omega}, \hat{\kappa} > := \text{tr}(\hat{\omega}^* \hat{\kappa}), \quad \hat{\omega}, \hat{\kappa} \in \pi(\Omega^p\mathcal{A}) \tag{3}$$

for elements of equal degree and by zero for two elements of different degree. With this scalar product $\Omega_\mathcal{D}\mathcal{A}$ is a subspace of $\pi(\Omega\mathcal{A})$, by definition orthogonal to the junk. As a subspace, $\Omega_\mathcal{D}\mathcal{A}$ inherits a scalar product which deserves a special name $(\ ,\ )$. It is given by

$$(\omega, \kappa) = <\hat{\omega}, P\hat{\kappa}>, \quad \omega, \kappa \in \Omega_\mathcal{D}^p\mathcal{A}$$

where P is the orthogonal projector in $\pi(\Omega\mathcal{A})$ onto the ortho-complement of J and $\hat{\omega}$ and $\hat{\kappa}$ are any representatives in the classes ω and κ. Again the scalar product vanishes for forms with different degree. For real algebras, all traces must be understood as real part of the trace.

In Yang-Mills models coupling constants appear as parameterization of the most general gauge invariant scalar product. In the same spirit, we want the most general scalar product on $\pi(\Omega\mathcal{A})$ compatible with the underlying algebraic structure. It is given by

$$< \hat{\omega}, \hat{\kappa} >_z := \operatorname{tr}(\hat{\omega}^* \hat{\kappa}\, z), \quad \hat{\omega}, \hat{\kappa} \in \pi(\Omega^p\mathcal{A}), \tag{4}$$

where z is a positive operator on $\mathcal{H}$, that commutes with $\rho(\mathcal{A})$, with the Dirac operator $\mathcal{D}$ and with the chirality operator χ,

$$\chi\psi_L = -\psi_L, \quad \chi\psi_R = +\psi_R.$$

A natural subclass of these scalar products is constructed with operators z in the image under ρ of the center of $\mathcal{A}$.

Since π is a homomorphism of involution algebras the product in $\Omega_\mathcal{D}\mathcal{A}$ is given by matrix multiplication followed by the projection P. The involution is given by transposition and complex conjugation, i.e. the dual with respect to the scalar product of the Hilbert space $\mathcal{H}$. Note that this scalar product admits no generalization. W. Kalau et al. [10] discuss the computation of the junk and of the differential for matrix algebras.

At this stage there is a first contact with gauge theories. Consider the vector space of anti-Hermitian 1-forms $\{H \in \Omega_\mathcal{D}^1\mathcal{A},\ H^* = -H\}$. A general element H is of the form

$$H = i \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

with h a finite sum of terms $\rho_L(a_0)[\rho_L(a_1)\mathcal{M} - \mathcal{M}\rho_R(a_1)] : \mathcal{H}_R \to \mathcal{H}_L,\ a_0, a_1 \in \mathcal{A}$. These elements are called Higgses or gauge potentials. In fact the space of gauge potentials carries an affine representation of the group of unitaries

$$G := \{g \in \mathcal{A},\ gg^* = g^*g = 1\}$$

defined by

$$\begin{aligned} H^g &:= \rho(g)H\rho(g^{-1}) + \rho(g)\delta(\rho(g^{-1})) \\ &= \rho(g)H\rho(g^{-1}) + (-i)\rho(g)[\mathcal{D}, \rho(g^{-1})] \\ &= \rho(g)[H - i\mathcal{D}]\rho(g^{-1}) + i\mathcal{D} \end{aligned}$$

$$= i \begin{pmatrix} 0 & h^g \\ (h^g)^* & 0 \end{pmatrix}$$

with $h^g - \mathcal{M} := \rho_L(g)[h - \mathcal{M}]\rho_R(g^{-1})$. H^g is the "gauge transformed of H". As usual every gauge potential H defines a covariant derivative $\delta + H$, covariant under the left action of G on $\Omega_{\mathcal{D}}\mathcal{A}$:

$$^g\omega := \rho(g)\omega, \quad \omega \in \Omega_{\mathcal{D}}\mathcal{A}$$

which means

$$(\delta + H^g)\,{}^g\omega = {}^g\left[(\delta + H)\omega\right].$$

Also we define the curvature C of H by

$$C := \delta H + H^2 \in \Omega_{\mathcal{D}}^2\mathcal{A}.$$

Note that here and later, H^2 is considered as element of $\Omega_{\mathcal{D}}^2\mathcal{A}$ which means it is the projection P applied to $H^2 \in \pi(\Omega^2\mathcal{A})$. The curvature C is a Hermitian 2-form with *homogeneous* gauge transformations

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g)C\rho(g^{-1}).$$

Finally, we define the preliminary Higgs potential $V_0(H)$, a functional on the space of gauge potentials, by

$$V_0(H) := (C, C) = \operatorname{tr}\left[(\delta H + H^2)P(\delta H + H^2)\right].$$

It is a polynomial of degree 4 in H with real, non-negative values. Furthermore it is gauge invariant, $V_0(H^g) = V_0(H)$, because of the homogeneous transformation property of the curvature C and because the orthogonal projector P commutes with all gauge transformations, $\rho(g)P = P\rho(g)$. The most remarkable property of the preliminary Higgs potential is that, in most cases, its vacuum spontaneously breaks the group G. To simplify the discussion, let us assume that the Dirac operator is a 1-form,

$$\mathcal{D} \in \Omega_{\mathcal{D}}^1\mathcal{A}. \tag{5}$$

Models not satisfying this hypothesis typically have degenerate vacua [4]. Then, we can introduce the change of variables

$$\Phi := H - i\mathcal{D} =: i \begin{pmatrix} 0 & \varphi \\ \varphi^* & 0 \end{pmatrix} \in \Omega_{\mathcal{D}}^1\mathcal{A} \tag{6}$$

with $\varphi = h - \mathcal{M}$. Assuming of course a gauge invariant internal Dirac operator, $\mathcal{D}^g = \mathcal{D}$, Φ is homogeneously transformed into

$$\Phi^g = H^g - \mathcal{D} = \rho(g)[H - i\mathcal{D}]\rho(g^{-1}) + i\mathcal{D} - i\mathcal{D} = \rho(g)\Phi\rho(g^{-1}), \tag{7}$$

and

$$\varphi^g = \rho_L(g)\varphi\rho_R(g^{-1}).$$

Now $h = 0$, or equivalently $\varphi = -\mathcal{M}$, is certainly a minimum of the preliminary Higgs potential and this minimum spontaneously breaks G if it is gauge variant and non-degenerate.

Consider two extreme classes of examples, vector-like and left-right models.

A *vector-like model* is defined by an arbitrary internal algebra $\mathcal{A}$ with identical left and right representations, $\rho_L = \rho_R$, and with a mass matrix proportional to the identity in each irreducible component. Since $\mathcal{D}$ and ρ commute, the internal differential algebra is trivial, $\Omega_{\mathcal{D}}^p \mathcal{A} = 0$ for $p \geq 1$ and the Dirac operator is not a 1-form, $\mathcal{D} \notin \Omega_{\mathcal{D}}^1 \mathcal{A}$. However, as we shall see, every vector-like model produces a Yang-Mills model with unbroken parity and unbroken gauge symmetry as electromagnetism or chromodynamics.

We define a *left-right model* by an internal algebra consisting of a sum of a "left-handed" and a "right-handed" algebra, $\mathcal{A} = \mathcal{A}_L \oplus \mathcal{A}_R$ with the left-handed algebra acting only on left-handed fermions and similarly for right-handed

$$\rho_L(a_L, a_R) = \rho_L(a_L, 0), \quad \rho_R(a_L, a_R) = \rho_R(0, a_R), \quad a_L \in \mathcal{A}_L, \ a_R \in \mathcal{A}_R.$$

Now, any non-vanishing fermion mass matrix breaks the gauge invariance. At the same time, the internal Dirac operator is always a 1-form, $\mathcal{D} \in \Omega_{\mathcal{D}}^1 \mathcal{A}$.

2.2 Adding spacetime

In the next step, the vectors ψ_L, ψ_R, and H are promoted to genuine fields, i.e. rendered spacetime dependent. As already in classical quantum mechanics, this is achieved by tensorizing with functions. Let us denote by $\mathcal{F}$ the algebra of (smooth, real or complex valued) functions over spacetime M. Consider the algebra $\mathcal{A}_t := \mathcal{F} \otimes \mathcal{A}$. The group of unitaries of the tensor algebra $\mathcal{A}_t$ is the gauged version of the group of unitaries of the internal algebra $\mathcal{A}$, i.e. the group of functions from spacetime into the group G. Consider the representation $\rho_t := \underline{\cdot} \otimes \rho$ of the tensor algebra on the tensor product $\mathcal{H}_t := \mathcal{S} \otimes \mathcal{H}$, where $\mathcal{S}$ is the Hilbert space of square integrable spinors on which functions act by multiplication: $(\underline{f}\psi)(x) := f(x)\psi(x)$, $f \in \mathcal{F}$, $\psi \in \mathcal{S}$. The definition of the tensor product of Dirac operators,

$$\mathcal{D}_t := \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes \mathcal{D}$$

comes from non-commutative geometry. We now repeat the above construction for the infinite dimensional algebra $\mathcal{A}_t$ and its K-cycle. As already stated, for $\mathcal{A} = \mathbb{C}$, $\mathcal{H} = \mathbb{C}$, $\mathcal{M} = 0$ the differential algebra $\Omega_{\mathcal{D}_t} \mathcal{A}_t$ is isomorphic to the de Rham algebra of differential forms $\Omega(M, \mathbb{C})$. For general $\mathcal{A}$, using the notations of [11], an anti-Hermitian 1-form $H_t \in \Omega_{\mathcal{D}_t}^1 \mathcal{A}_t$,

$$H_t = A + H,$$

contains two pieces, an anti-Hermitian Higgs *field* $H \in \Omega^0(M, \Omega^1_{\mathcal{D}}\mathcal{A})$ and a genuine gauge field $A \in \Omega^1(M, \rho(\mathfrak{g}))$ with values in the Lie algebra of the group of unitaries, $\mathfrak{g} := \{X \in \mathcal{A}, \ X^* = -X\}$, represented on $\mathcal{H}$. The curvature of H_t

$$C_t := \delta_t H_t + H_t^2 \ \in \Omega^2_{\mathcal{D}}\mathcal{A}_t$$

contains three pieces,

$$C_t = C + F - \mathrm{D}\Phi\gamma_5,$$

the ordinary, now x-dependent, curvature $C = \delta H + H^2$, the field strength

$$F := \mathrm{d}A + \frac{1}{2}[A, A] \quad \in \Omega^2(M, \rho(\mathfrak{g}))$$

and the covariant derivative of Φ

$$\mathrm{D}\Phi = \mathrm{d}\Phi + [A, \Phi] \quad \in \Omega^1(M, \Omega^1_{\mathcal{D}}\mathcal{A}).$$

Note that the covariant derivative may be applied to Φ thanks to its homogeneous transformation law, equation (7).

The definition of the Higgs potential in the infinite dimensional space

$$V_t(H_t) := (C_t, C_t)$$

requires a suitable regularisation of the sum of eigenvalues over the space of spinors $\mathcal{S}$. Here, we have to suppose spacetime to be compact and Euclidean. Then the regularisation is achieved by the Dixmier trace which allows an explicit computation of V_t. One of the miracles in CL models is that V_t alone reproduces the complete bosonic action of a YMH model. Indeed it consists of three pieces, the Yang-Mills action, the covariant Klein-Gordon action and an integrated Higgs potential

$$V_t(A + H) = \int_M \mathrm{tr}\,(F * F\, z) + \int_M \mathrm{tr}\,(\mathrm{D}\Phi^* * \mathrm{D}\Phi\, z) + \int_M *V(H). \tag{8}$$

As the preliminary Higgs potential V_0, the (final) Higgs potential V is calculated as a function of the fermion masses,

$$V := V_0 - \mathrm{tr}\,[\alpha C^* \alpha C\, z] = \mathrm{tr}\,[(C - \alpha C)^*(C - \alpha C)\, z].$$

The linear map $\alpha : \Omega^2_{\mathcal{D}}\mathcal{A} \longrightarrow \rho(\mathcal{A}) + \pi(\delta(\ker \pi)^1)$ is determined by the two equations

$$\mathrm{tr}\,[R^*(C - \alpha C)\, z] = 0 \qquad \text{for all } R \in \rho(\mathcal{A}), \tag{9}$$

$$\mathrm{tr}\,[K^* \alpha C\, z] = 0 \qquad \text{for all } K \in \pi(\delta(\ker \pi)^1). \tag{10}$$

All remaining traces are over the finite dimensional Hilbert space $\mathcal{H}$.

Another miracle happens in the fermionic sector, where the completely covariant action $\psi^*(\mathcal{D}_t + iH_t)\psi$ reproduces the complete fermionic action of a YMH model. We denote by

$$\psi = \psi_L + \psi_R \ \in \mathcal{H}_t = \mathcal{S} \otimes (\mathcal{H}_L \oplus \mathcal{H}_R)$$

the multiplets of spinors and by ψ^* the dual of ψ with respect to the scalar product of the concerned Hilbert space. Then

$$
\begin{aligned}
\psi^*(\mathcal{D}_t + iH_t)\psi &= \int_M *\psi^*(\not{\partial} + i\gamma(A))\psi - \int_M *(\psi_L^* h \gamma_5 \psi_R + \psi_R^* h^* \gamma_5 \psi_L) \\
&\quad + \int_M *(\psi_L^* \mathcal{M} \gamma_5 \psi_R + \psi_R^* \mathcal{M}^* \gamma_5 \psi_L) \\
&= \int_M *\psi^*(\not{\partial} + i\gamma(A))\psi - \int_M *(\psi_L^* \varphi \gamma_5 \psi_R + \psi_R^* \varphi^* \gamma_5 \psi_L) \quad (11)
\end{aligned}
$$

containing the ordinary Dirac action and the Yukawa couplings. If the minimum $\varphi = v$ is non-degenerate, we retrieve the input fermionic mass matrix $\mathcal{M}$ on the output side by setting the perturbative variables h to zero in the first equation in (11). The rhs of the second equation in (11) is the fermionic action written with the homogeneous scalar variable φ. The second term yields the trilinear invariants (1) with Yukawa couplings fixed such that $\mathcal{M}$ is the fermionic mass matrix. Consequently every CL model is a YMH model with $\mathcal{H}_S = \{H \in \Omega_{\mathcal{D}}^1 \mathcal{A},\ H^* = -H\}$. Note that $\mathcal{H}_S$ carries a group representation, that is not necessarily an algebra representation and we have the following inclusion of group representations $\mathcal{H}_S \subset (\mathcal{H}_L^* \otimes \mathcal{H}_R) \oplus (\mathcal{H}_R^* \otimes \mathcal{H}_L)$.

3 Necessary conditions

One may very well do general relativity using only Euclidean geometry. Still, we agree that Riemannian geometry is the natural setting of general relativity. A main argument in favor of this attitude is that there are infinitely more gravitational theories in Euclidean geometry than in Riemannian geometry. The same is true for the standard model. Its natural setting, to our taste, is non-commutative geometry. The fact that today's Yang-Mills-Higgs model of electro-weak and strong interactions falls in the infinitely smaller class of Connes-Lott models is remarkable. The purpose of this section is to show in what extent it is remarkable. We give a list of constraints on the input of a YMH model. They are necessary conditions for the existence of a corresponding CL model.

3.1 The group

The compact Lie group G defining a Yang-Mills model must be chosen such that its Lie algebra $\mathfrak{g}$ admits an invariant scalar product. Therefore $\mathfrak{g}$ is a direct sum of simple and abelian algebras. After complexification, the simple Lie algebras are classified according to E. Cartan, into four infinite series, $su(n+1),\ n \geq 1,\quad o(2n+1),\ n \geq 2,\quad sp(n),\ n \geq 3,\quad o(2n),\ n \geq 4$ and

five exceptional algebras G_2, F_4, E_6, E_7, E_8. To define a CL model, we need a real or complex involution algebra $\mathcal{A}$ admitting a finite dimensional, faithful representation. Their classification is easy. In the complex case, such an algebra is a direct sum of matrix algebras $M_n(\mathbb{C})$, $n \geq 1$. In the real case, we have direct sums of matrix algebras with real, complex or quaternionic coefficients, $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{H})$, $n \geq 1$. The corresponding groups of unitaries are $O(n, \mathbb{R})$, $U(n)$, $USp(n)$. Note the two isomorphisms, $USp(2) \cong SU(2)$ and $USp(4)/\mathbb{Z}_2 \cong SO(5, \mathbb{R})$.

The groups accessible in a CL model therefore belong to the second, third, and forth Cartan series. Furthermore we have $u(n) \cong su(n) \oplus u(1)$. Up to the $u(1)$ factor, this is the first series. At the group level, this factor is disposed of by a condition on the determinant. In the algebraic setting there is a similar condition, that reduces the group of unitaries to a subgroup, here $SU(n)$. This condition is called unimodularity and is discussed in the next section. To sum up, all classical Lie groups are accessible in a CL model but the exceptional ones.

3.2 The fermion representation

In a YMH model, the left- and right-handed fermions come in unitary representations of the chosen group G. Every G has an infinite number of irreducible, unitary representations. They are classified by their maximal weight. On the other hand, the above involution algebras $\mathcal{A}$ admit only one or two irreducible representations. The reason is that an algebra representation has to respect the multiplication and the linear structure, while a group representation has to respect only the multiplication. In particular, the tensor product of two group representations is a group representation, while the tensor product of two algebra representations is not an algebra representation, in general.

The only irreducible representation of $M_n(\mathbb{C})$ as complex algebra is the fundamental one on $\mathcal{H} = \mathbb{C}^n$. Also $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ have only the fundamental representations on $\mathcal{H} = \mathbb{R}^n$ and $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^2$ while $M_n(\mathbb{C})$ considered as real algebra has two inequivalent, irreducible representations, the fundamental one: $\mathcal{H} = \mathbb{C}^n$, $\rho_1(a) = a$, $a \in M_n(\mathbb{C})$, and its conjugate: $\mathcal{H} = \mathbb{C}^n$, $\rho_2(a) = \bar{a}$.

Therefore, the only possible representations for fermions in a CL model are

- for $G = O(n, \mathbb{R})$, N generations of the fundamental representation on $\mathcal{H} = \mathbb{R}^n \otimes \mathbb{R}^N$,

- for $G = U(n)$ (or $SU(n)$), N generations of the fundamental representation on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^N$ and $\bar{N}$ generations of its conjugate on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^{\bar{N}}$,

- for $G = USp(n)$, N generations of the fundamental representation on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$.

In a YMH model with $G = SU(2)$, the fermions can be put in any irreducible representations of dimension 1, 2, 3,... while in the corresponding CL model with $\mathcal{A} = \mathbb{H}$, there is only one irreducible representation accessible for the fermions, the fundamental, two dimensional one.

Similarly, in a YMH model with $G = U(1)$ the fermions can have any (electric) charge from $\mathbb{Z}$ or even from $\mathbb{R}$ if we allow 'spinor' representations. In the corresponding CL model with $\mathcal{A} = \mathbb{C}$, fermions can only have charges plus and minus one. In any case, if we want more fermions in a CL model, we are forced to introduce families of fermions.

At this point, we realize that all popular grand unified models are excluded by Connes-Lott.

3.3 The gauge coupling constants

In a YMH model, the gauge coupling constants parameterize the most general gauge invariant scalar product on the Lie algebra $\mathfrak{g}$ of G. In a CL model, see the rhs of equation (8), this scalar product is not general but comes from the trace over the fermion representation space $\mathcal{H}$, equation (4). The scalar product involves the positive operator z, that commutes with the internal Dirac operator and with the fermion transformations $\rho(\mathcal{A})$ and that leaves $\mathcal{H}_L$ and $\mathcal{H}_R$ invariant. Depending on the details of the mass matrix and of the left- and right-handed representations ρ_L and ρ_R, the gauge coupling constants may be constraint or not.

3.4 The Higgs sector

As explained in section 2, the scalar representation ρ_S on $\mathcal{H}_S$ in a CL model is a representation of the *group* of unitaries only. This representation is not chosen but it is calculated as a function of the left- and right-handed fermion representations and of the mass matrix. The dependence of the scalar representation on this input is involved and we can make only one general statement:

$$\mathcal{H}_S \subset (\mathcal{H}_L^* \otimes \mathcal{H}_R) \oplus (\mathcal{H}_R^* \otimes \mathcal{H}_L)$$

which implies that the invariance group of the fermionic mass terms is equal to the unbroken subgroup. In a general YMH model the latter is only a subgroup of the former, e.g. minimal $SU(5)$. Also, this inclusion is sufficient to rule out the possibility of spontaneous parity breaking in left-right symmetric models à la Connes-Lott [12].

The Higgs potential as well, is on the output side of a CL model. Its calculation involves the positive operator z from the input and is by far, the most complicated calculation in this scheme. We know that $\varphi = -\mathcal{M}$ is an absolute minimum of the Higgs potential. If it is non-degenerate, the gauge and scalar boson masses are determined by the fermion masses and the entries of z.

Our last necessary condition concerns the Yukawa couplings. In a CL model, they are determined such that $\mathcal{M}$ is the fermionic mass matrix after spontaneous symmetry breaking. Up to the z dependent scalar normalization in the bosonic action (8), the Yukawa couplings are all one.

4   The unimodularity condition

The purpose of the unimodularity condition is to reduce the group of unitaries $U(n)$ to its subgroup $SU(n)$. At the group level, this is easily achieved by the condition $\det g = 1$. However the determinant being a non-linear function is not available at the algebra level. We are lead to use the trace instead, together with the formula

$$\det e^{2\pi i X} \;=\; e^{2\pi i \,\mathrm{tr}\, X}.$$

Even in the infinite dimensional case, the connected component G^0 of the unit in the group of unitaries G is generated by elements $g = e^{2\pi i X}$, $X = X^* \in \mathcal{A}$. The desired reduction is achieved by using the phase, defined by

$$\mathrm{phase}_\tau(g) := \frac{1}{2\pi i} \int_0^1 \tau\left(g(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t)^{-1}\right)\,\mathrm{d}t,$$

where τ is a linear form on $\mathcal{A}$ satisfying

$$\tau(1) \in \mathbb{Z}, \quad \tau(a^*) = \tau(a)^*, \quad \tau(a) = \tau(g^*ag), \quad g \in G,\ a \in \mathcal{A}^+ := \{bb^*,\ b \in \mathcal{A}\},$$

and where $g(t)$ is a curve in G^0 connecting the unit to g. We obtain the finite dimensional case above by putting $\tau(a) = \mathrm{tr}\, \rho(a)$ and $g(t) = e^{2\pi i X t}$. The definition of the phase involves two choices, that are easily controlled in finite dimensions: the most general linear form τ can be written as $\tau(a) = \mathrm{tr}\, \rho(ap)$, $a \in \mathcal{A}$, $p \in \mathrm{center}\,\mathcal{A}$, and the ambiguity in the choice of the curve $g(t)$ is controlled by the first fundamental group $\pi^1(G^0)$ which is contained in $\mathbb{Z}$, see table below. Therefore the unimodularity condition

$$e^{2\pi i \,\mathrm{phase}_\tau(g)} = 1$$

is well defined and defines a subgroup

$$G_p := \left\{g \in G^0,\ e^{2\pi i \,\mathrm{phase}_{\mathrm{tr}\,\rho(\cdot p)}(g)} = 1\right\}$$

of G^0. For $\mathcal{A} = M_n(\mathbb{C})$, $n \geq 2$, the center is spanned by 1_n and $G_1 = SU(n)$. The center of $\mathcal{A} = M_n(\mathbb{C}) \oplus M_m(\mathbb{C})$, $n, m \geq 2$, is spanned by two elements, p_n and p_m, the projectors on $M_n(\mathbb{C})$ and on $M_m(\mathbb{C})$. We have

$$
\begin{aligned}
G_{p_n} &= SU(n) \times U(m),\\
G_{p_m} &= U(n) \times SU(m),\\
G_{p_n+p_m} &= S(U(n) \times U(m)).
\end{aligned}
$$

5   The standard model à la Connes-Lott

5.1 Input

The standard model in non-commutative geometry is described by two real algebras, a left-right one for electro-weak interactions: $\mathcal{A} := \mathbb{H} \oplus \mathbb{C}$ with group of unitaries $SU(2) \times U(1)$, and a vector-like one for strong interactions: $\mathcal{A}' := M_3(\mathbb{C}) \oplus \mathbb{C}$ with group of unitaries $U(3) \times U(1)$. We denote by $\mathbb{H}$ the algebra of quaternions. Its elements are complex 2×2 matrices of the form

$$\begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}, \quad x, y \in \mathbb{C}.$$

Both algebras $\mathcal{A}$ and $\mathcal{A}'$ are represented on the same Hilbert space $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$ of left- and right-handed fermions,

$$\mathcal{H}_L = \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^N\right),$$

$$\mathcal{H}_R = \left((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C} \otimes \mathbb{C}^N\right).$$

The first factor denotes weak isospin, the second N generations, $N = 3$, and the third denotes colour triplets and singlets. With respect to the standard basis

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$$

of $\mathcal{H}_L$ and

$$\begin{matrix} u_R, & c_R, & t_R, \\ d_R, & s_R, & b_R, \end{matrix} \quad e_R, \quad \mu_R, \quad \tau_R$$

of $\mathcal{H}_R$, the representations are given by block diagonal matrices. For $(a, b) \in \mathbb{H} \oplus \mathbb{C}$ we set

$$B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}$$

and define a representation of $\mathcal{A}$ by

$$\rho(a, b) := \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & a \otimes 1_N & 0 & 0 \\ 0 & 0 & B \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & \bar{b}1_N \end{pmatrix} = \begin{pmatrix} \rho_L(a) & 0 \\ 0 & \rho_R(b) \end{pmatrix}$$

and for $(c, d) \in M_3(\mathbb{C}) \oplus \mathbb{C}$ we define a $\mathcal{A}'$ representation

$$\rho'(c, d) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 & 0 & 0 \\ 0 & d1_2 \otimes 1_N & 0 & 0 \\ 0 & 0 & 1_2 \otimes 1_N \otimes c & 0 \\ 0 & 0 & 0 & d1_N \end{pmatrix}.$$

The last piece of input is the fermion mass matrix $\mathcal{M}$ which constitutes the self adjoint 'internal Dirac operator':

$$\mathcal{D} := \begin{pmatrix} 0 & 0 & \begin{pmatrix} M_u \otimes 1_3 & 0 \\ 0 & M_d \otimes 1_3 \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \\ \begin{pmatrix} M_u^* \otimes 1_3 & 0 \\ 0 & M_d^* \otimes 1_3 \end{pmatrix} & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & M_e^* \end{pmatrix} & 0 & 0 \end{pmatrix}$$

$$=: \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^* & 0 \end{pmatrix}$$

with

$$M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d := C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}$$

where C_{KM} denotes the Cabbibo-Kobayashi-Maskawa matrix. All indicated fermion masses are supposed positive and different. Note that the strong interactions are vector-like: the chirality operator

$$\chi = \begin{pmatrix} -1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & -1_2 \otimes 1_N & 0 & 0 \\ 0 & 0 & 1_2 \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & 1_N \end{pmatrix}$$

and the 'Dirac operator' commute with $\mathcal{A}'$, $[\mathcal{D}, \rho'(\mathcal{A}')] = 0$, $[\chi, \rho'(\mathcal{A}')] = 0$.

5.2   Internal space

We now apply the construction outlined above to the standard model. Obviously, the standard model is not the right example to get familiar with the Connes-Lott scheme. Miraculously enough, the standard model contains the minimax example, analogue of the Georgi-Glashow $SO(3)$ model [13] in the Yang-Mills-Higgs scheme (a maximum of pleasure with a minimum of effort). This example represents the electro-weak algebra $\mathcal{A} = \mathbb{H} \oplus \mathbb{C}$ on *two* generations of leptons. Its only drawback are neutrinos with electric charge, a drawback, that can be corrected by adding strong interactions.

Anyway, let us start the computation of the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ for the electro-weak algebra with generic element $(a, b) \in \mathbb{H} \oplus \mathbb{C}$ represented on the long list of fermions. A general 1-form is a sum of terms

$$\pi((a_0, b_0)\delta(a_1, b_1)) = -i \begin{pmatrix} 0 & \rho_L(a_0)(\mathcal{M}\rho_R(b_1) - \rho_L(a_1)\mathcal{M}) \\ \rho_R(b_0)(\mathcal{M}^*\rho_L(a_1) - \rho_R(b_1)\mathcal{M}^*) & 0 \end{pmatrix}$$

and as vector space

$$\Omega_{\mathcal{D}}^1 \mathcal{A} = \left\{ i \begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(\tilde{h}^*) & 0 \end{pmatrix}, \ h, \tilde{h} \in \mathbb{H} \right\}.$$

The Higgs being an anti-Hermitian 1-form

$$H = i \begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(h^*) & 0 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix} \in \mathbb{H}$$

is parameterized by one complex doublet

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathbb{C}.$$

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The junk in degree two turns out to be

$$J^2 = \left\{ i \begin{pmatrix} j \otimes \Delta & 0 \\ 0 & 0 \end{pmatrix}, \quad j \in \mathbb{H} \right\}$$

with

$$\Delta := \frac{1}{2} \begin{pmatrix} (M_u M_u^* - M_d M_d^*) \otimes 1_3 & 0 \\ 0 & -M_e M_e^* \end{pmatrix}.$$

To project it out, we use the general scalar product (4) with the real part of the trace. Without loss of generality [3], we may immediately use a z, that commutes with $\rho(\mathcal{A})$ *and* with $\rho(\mathcal{A}')$ and of course with $\mathcal{D}$ and χ. It has the form

$$z = \begin{pmatrix} x/3 \, 1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2 \otimes y & 0 & 0 \\ 0 & 0 & x/3 \, 1_2 \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \tag{12}$$

where y is a positive, diagonal $N \times N$ matrix and x is a positive number. The scalar product defined with this z has a natural interpretation. Indeed, the general scalar product is just a sum of the simplest scalar products in each irreducible part of the fermion representation, each poised with a separate positive constant. We have four irreducible parts, the three lepton families and all quarks together. Due to the Cabbibo-Kobayashi-Maskawa mixing, the ponderations of the three quark families are identical. If, in addition, we suppose that z lie in $\rho(\text{center}\mathcal{A})$ then we have, $y = \lambda 1_N$ with a positive constant λ.

With respect to the general scalar product, we can write the 2-forms as

$$\Omega_{\mathcal{D}}^2 \mathcal{A} = \left\{ \begin{pmatrix} \tilde{c} \otimes \Sigma & 0 \\ 0 & \mathcal{M}^* \rho_L(c) \mathcal{M} \end{pmatrix}, \quad \tilde{c}, c \in \mathbb{H} \right\}$$

with

$$\Sigma := \frac{1}{2} \begin{pmatrix} (M_u M_u^* + M_d M_d^*) \otimes 1_3 & 0 \\ 0 & M_e M_e^* \end{pmatrix}.$$

Since π is a homomorphism of involution algebras, the product in $\Omega_{\mathcal{D}} \mathcal{A}$ is given by matrix multiplication followed by the orthogonal projection P and the involution is given by transposition and complex conjugation. In order to calculate the differential δ, we go back to the universal differential envelope. The result is

$$\delta : \Omega_{\mathcal{D}}^1 \mathcal{A} \longrightarrow \Omega_{\mathcal{D}}^2 \mathcal{A}$$

$$i \begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^* \rho_L(\tilde{h}^*) & 0 \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{c} \otimes \Sigma & 0 \\ 0 & \mathcal{M}^* \rho_L(c)\mathcal{M} \end{pmatrix}$$

with

$$\tilde{c} = c = h + \tilde{h}^*.$$

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We are now in position to compute the curvature and the preliminary Higgs potential:

$$C := \delta H + H^2 = \left(1 - |\varphi|^2\right) \begin{pmatrix} 1_2 \otimes \Sigma & 0 \\ 0 & \mathcal{M}^*\mathcal{M} \end{pmatrix}$$

where we have introduced the homogeneous scalar variable

$$\Phi := H - i\mathcal{D} =: i \begin{pmatrix} 0 & \rho_L(\varphi)\mathcal{M} \\ \mathcal{M}^*\rho_L(\varphi^*) & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix} \in \mathbb{H},$$

$$|\varphi|^2 := |\varphi_1|^2 + |\varphi_2|^2.$$

The preliminary Higgs potential

$$V_0 = \operatorname{tr}\left[C^2\right] = \left(1 - |\varphi|^2\right)^2 \times \left(\frac{2}{3}\operatorname{tr}\left[(M_u^*M_u)^2\right]x + \frac{2}{3}\operatorname{tr}\left[(M_d^*M_d)^2\right]x\right.$$

$$+ \frac{1}{2}\operatorname{tr}\left[M_u^*M_u M_d^*M_d\right]x + \frac{1}{2}\operatorname{tr}\left[M_d^*M_d M_u^*M_u\right]x$$

$$\left.+ \frac{2}{3}\operatorname{tr}\left[(M_e^*M_e)^2 y\right]\right)$$

breaks the $SU(2) \times U(1)$ symmetry down to $U(1)$.

Finally the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}'$ of the strong algebra is trivial because strong interactions are vector-like.

5.3 Adding spacetime

Recall the expression of the curvature in the electro-weak sector

$$C = \left(1 - |\varphi|^2\right) \begin{pmatrix} 1_2 \otimes \Sigma & 0 \\ 0 & \mathcal{M}^*\mathcal{M} \end{pmatrix}.$$

A straightforward application of equations (9, 10) — taking the real part of the traces is understood — yields the projection αC. It is again block diagonal with diagonal elements:

$$\alpha C_{qL} = \frac{1 - |\varphi|^2}{2} \frac{\operatorname{tr}\left[M_u^*M_u\right]x + \operatorname{tr}\left[M_d^*M_d\right]x + \operatorname{tr}\left[M_e^*M_e\, y\right]}{Nx + \operatorname{tr} y} 1_2 \otimes 1_N \otimes 1_3$$

$$\alpha C_{\ell L} = \frac{1 - |\varphi|^2}{2} \frac{\operatorname{tr}\left[M_u^*M_u\right]x + \operatorname{tr}\left[M_d^*M_d\right]x + \operatorname{tr}\left[M_e^*M_e\, y\right]}{Nx + \operatorname{tr} y} 1_2 \otimes 1_N$$

$$\alpha C_{qR} = \frac{1 - |\varphi|^2}{2} \frac{\operatorname{tr}\left[M_u^*M_u\right]x + \operatorname{tr}\left[M_d^*M_d\right]x + \operatorname{tr}\left[M_e^*M_e\, y\right]}{Nx + \operatorname{tr} y/2} 1_2 \otimes 1_N \otimes 1_3$$

$$\alpha C_{\ell R} = \frac{1 - |\varphi|^2}{2} \frac{\operatorname{tr}\left[M_u^*M_u\right]x + \operatorname{tr}\left[M_d^*M_d\right]x + \operatorname{tr}\left[M_e^*M_e\, y\right]}{Nx + \operatorname{tr} y/2} 1_N.$$

The Higgs potential is computed next,

$$V = K \left(1 - |\varphi|^2\right)^2,$$

$$
\begin{aligned}
K \quad := \quad & \frac{3}{2}\,\mathrm{tr}\,\left[(M_u^* M_u)^2\right] x + \frac{3}{2}\,\mathrm{tr}\,\left[(M_d^* M_d)^2\right] x \\
& + \mathrm{tr}\,\left[M_u^* M_u M_d^* M_d\right] x \\
& + \frac{3}{2}\,\mathrm{tr}\,\left[M_e^* M_e M_e^* M_e\, y\right] \\
& - \frac{1}{2} L^2 \left[\frac{1}{Nx + \mathrm{tr}\,y} + \frac{1}{Nx + \mathrm{tr}\,y/2}\right],
\end{aligned}
\tag{13}
$$

$$
L := \mathrm{tr}\,\left[M_u^* M_u\right] x + \mathrm{tr}\,\left[M_d^* M_d\right] x + \mathrm{tr}\,\left[M_e^* M_e\, y\right]. \tag{14}
$$

Note that the scalar fields φ_1 and φ_2 are not properly normalized, they are dimensionless. To get their normalization straight we have to compute the factor in front of the kinetic term in the Klein-Gordon action:

$$
\mathrm{tr}\,\left(\,\mathrm{d}\Phi^* * \mathrm{d}\Phi\, z\right) = *2L|\partial\varphi|^2.
$$

Likewise, we need the normalization (cf. appendix of [4]) of the electro-weak gauge bosons:

$$
\mathrm{tr}\,\left(F * F\, z\right) = *E\left(\partial_\mu W_\nu^+ \, \partial^\mu W^{-\nu} - ...\right)
$$

with

$$
E := Nx + \mathrm{tr}\,y. \tag{15}
$$

We end up with the following masses:

$$
m_W^2 \quad = \quad \frac{L}{E}, \tag{16}
$$

$$
m_H^2 \quad = \quad \frac{2K}{L}. \tag{17}
$$

Finally, we turn to the relations among coupling constants. As already pointed out, they are due to the fact that the gauge invariant scalar product on the internal Lie algebra, the Lie algebra of the group of unitaries $\mathfrak{g} := \{X \in \mathcal{A},\ X^* + X = 0\}$, in the Yang-Mills action (8) is not general but stems from the trace over the fermion representation ρ on $\mathcal{H}$. Since this representation is faithful the scalar product (3) indeed induces an invariant scalar product on $\mathfrak{g}$. In the case at hand, our Lie algebra is a direct sum $\mathfrak{g} \oplus \mathfrak{g}'$. We define the invariant scalar product by

$$
(X_1 + X_1', X_2 + X_2') := \mathrm{tr}\,\left[\rho(X_1)^* \rho(X_2)\, z\right] + \mathrm{tr}\,\left[\rho(X_1')^* \rho(X_2')\, z'\right]
$$

where z given by equation (12) and z' are two *independent* elements in the intersection of the commutants of $\rho(\mathcal{A})$ and $\rho'(\mathcal{A}')$,

$$
z' = \begin{pmatrix}
x'/3\, 1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\
0 & 1_2 \otimes y' & 0 & 0 \\
0 & 0 & x'/3\, 1_2 \otimes 1_N \otimes 1_3 & 0 \\
0 & 0 & 0 & y'
\end{pmatrix},
$$

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y' is a positive, diagonal $N \times N$ matrix and x' a positive number.

The fact that the standard model can be written in the setting of non-commutative geometry depends crucially, at this point, on two happy circumstances. Firstly, the electric charge 'generator'

$$Q = \begin{pmatrix} \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1_N & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & -1_N \end{pmatrix}$$

is an element of $i\rho(\mathfrak{g}) \oplus i\rho'(\mathfrak{g}')$. Indeed it is a linear combination of weak isospin I_3 and elements of the three $u(1)$ factors:

$$Q = \rho \left(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, 0 \right) + \frac{1}{2i}\rho(0, i) + \frac{1}{6i}\rho'(i1_3, 0) - \frac{1}{2i}\rho'(0, i).$$

We have put 'generator' in quotation marks because iQ is a Lie algebra element, not Q. The weak angle θ_w measures the proportion of weak isospin in the electric charge:

$$\frac{Q}{|Q|} = \sin\theta_w \frac{I_3}{|I_3|} + \cos\theta_w \frac{Y}{|Y|}. \tag{18}$$

The hypercharge Y is a linear combination of the three $u(1)$ factors,

$$Y := \frac{1}{2i}\rho(0, i) + \frac{1}{6i}\rho'(i1_3, 0) - \frac{1}{2i}\rho'(0, i).$$

Here comes the second happy circumstance, this particular combination Y is singled out by two unimodularity conditions. They reduce the group of unitaries $SU(2) \times U(1) \times U(3) \times U(1)$ to $SU(2) \times U(1) \times SU(3)$ with the surviving $U(1)$ generated by the hypercharge. Indeed, the center of $\mathcal{A} \oplus \mathcal{A}'$ is four dimensional with basis $p_1, ..., p_4$. $p_1 := \rho(1_2, 0)$ projects on $\mathbb{H}$, $p_2 := \rho(0, 1)$ on $\mathbb{C}$, $p_3 := \rho'(1_3, 0)$ on $M_3(\mathbb{C})$, and $p_4 = \rho'(0, 1)$ on $\mathbb{C}'$, and the group of the standard model is $G_{p_1} \cap G_{p_2}$.

Let us come back to the calculation of the weak angle. Equation (18) is a matrix of equations. Let us take the difference of the two diagonal elements corresponding to the left handed neutrino and electron:

$$\frac{1}{|Q|} = \sin\theta_w \frac{1}{|I_3|},$$

$$\sin^2\theta_w = \frac{(I_3, I_3)}{(Q, Q)}.$$

The numerator is readily computed,

$$(I_3, I_3) = \mathrm{tr}\left[\rho \left(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, 0 \right)^2 z \right] = \frac{1}{2}(Nx + \mathrm{tr}\, y).$$

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We compute the denominator with Pythagoras' kind help,

$$
\begin{aligned}
(Q,Q) &= \operatorname{tr}\left[\rho\left(\left(\begin{array}{cc} 1/2 & 0 \\ 0 & -1/2 \end{array}\right),0\right)^2 z\right] + \frac{1}{4}\operatorname{tr}\left[\rho(0,1)^2 z\right] \\
&\quad + \frac{1}{36}\operatorname{tr}\left[\rho'(1_3,0)^2 z'\right] + \frac{1}{4}\operatorname{tr}\left[\rho'(0,1)^2 z'\right] \\
&= \left(Nx + \frac{3}{4}\operatorname{tr}y\right) + \frac{1}{3}x' + \frac{3}{4}\operatorname{tr}y'.
\end{aligned}
$$

Finally the mixing angle is given by

$$
\sin^2\theta_w = \frac{Nx + \operatorname{tr}y}{2Nx + \frac{3}{2}\operatorname{tr}y + \frac{2}{3}x' + \frac{3}{2}\operatorname{tr}y'}. \tag{19}
$$

In a similar fashion, the ratio between strong and weak coupling is computed,

$$
\left(\frac{g_3}{g_2}\right)^2 = \frac{(I_3,I_3)}{(C,C)} = \frac{1}{4}\frac{Nx + \operatorname{tr}y}{x'} \tag{20}
$$

where

$$
C := \rho'\left(\left(\begin{array}{ccc} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right),0\right).
$$

Here C stands for colour not for curvature.

In this calculation z and z' are different in general, implying that the electro-weak sector $\rho(\mathcal{A})$ is orthogonal to the strong sector $\rho'(\mathcal{A}')$. In the special case where $z = z'$ a different choice is possible:

$$
(a,a') := \operatorname{tr}\left[\rho(a)^*\rho'(a')z\right], \quad a \in \mathcal{A},\ a' \in \mathcal{A}'.
$$

Then the two $U(1)$ factors $\rho(0,1)$ and $\rho'(0,1)$ are not orthogonal anymore and the value of $\sin^2\theta_w$ comes out smaller [14]. This choice is closer to grand unified models and yields $\sin^2\theta_w = 3/8 = 0.375$ for $z = z' = 1$ to be compared to $\sin^2\theta_w = 12/29 = 0.414$ from equation (19).

6 Fuzzy relations

Non-commutative geometry produces relations among gauge couplings and boson and fermion masses. The aim of this section is a detailed study of these relations for the standard model. Here, we will encounter a new phenomenon, that we call fuzzy relations. To get a feeling for this phenomenon, it will be helpful to consider first simpler models. We start by switching off strong interactions. Indeed, since they are vector-like, they do not yet play an important role in the non-commutative setting.

So let us consider the real algebra $\mathbb{H} \oplus \mathbb{C}$ with N generations of leptons. In this case equations (15, 13, 14) reduce to

$$
E = \sum_{j=1}^{N} y_j,
$$

$$K = \frac{3}{2} \sum m_j^4 y_j - \frac{1}{2} L^2 \left[\frac{1}{\sum y_j} + \frac{2}{\sum y_j} \right],$$

$$L = \sum m_j^2 y_j,$$

where the y_j denote the eigenvalues of the positive diagonal matrix y. Recall that they are arbitrary positive parameters. Let us also recall the expressions (16, 17) of the W and Higgs masses $m_W^2 = L/E$, $m_H^2 = 2K/L$. Since only squares of masses appear, we will alleviate notations by putting

$$m_W^2 =: W, \quad m_H^2 =: H, \quad m_\tau^2 =: \tau, \quad \ldots$$

Now, one generation, $N = 1$, has a degenerate vacuum, i.e. a vanishing Higgs potential, $K = 0$. For more than one generations, this degeneracy is lifted.

If we take two generations, say $\mu < \tau$, we can eliminate the two positive unknown y_j from equations (16, 17) and we obtain the *exact* mass relation

$$H = 3 \left(\tau - W \right) \left(1 - \frac{\mu}{W} \right) \tag{21}$$

with $\mu < W < \tau$. This curve in the $m_\tau \, m_H$ plane is again a degenerate situation in the sense that with $N = 3$ generations (or more), $e < \mu < \tau$, this curve will become a band of width

$$\sqrt{\frac{m_\mu^2 - m_e^2}{m_W^2}} \sqrt{3 \left(m_\tau^2 - m_W^2 \right)}.$$

This is what we call a *fuzzy* mass relation.

Here are the details for $N = 3$. Equations (16, 17) are homogeneous in our three positive unknowns y_j. Therefore we introduce

$$z_1 := y_1/y_3, \quad z_2 := y_2/y_3,$$

and solve equation (16) with respect to z_2,

$$z_2 = \frac{\tau - W - z_1(W - e)}{W - \mu},$$

Eliminating z_2 from equation (17) we get

$$H/3 + W = \frac{-z_1(\mu - e)[\mu + e - \mu e/W] + (\tau - \mu)[\tau + \mu - \tau\mu/W]}{-z_1(\mu - e) + (\tau - \mu)}. \tag{22}$$

From equation (16), we know that the W mass lies between the masses of the lightest and of the heaviest lepton, $e < W < \tau$. Therefore, we have to distinguish two cases, $\mu < W$ and $\mu > W$. In the first case, as z_2 is positive, z_1 varies in a finite interval

$$0 < z_1 < \frac{\tau - W}{W - e}. \tag{23}$$

21

On the other hand, one checks easily that the rhs of equation (22) is an increasing function of z_1, and the inequalities (23) imply the inequalities

$$m_H(m_\tau; m_\mu) < m_H < m_H(m_\tau; m_e), \tag{24}$$

where we introduced the parameterized family of curves in the $m_\tau\, m_H$ plane

$$m_H(m_\tau; m) := \sqrt{3\,(m_\tau^2 - m_W^2)\,(1 - m^2/m_W^2)}.$$

The parameter m varies in the open interval $(0, m_W)$. All values of m_H in the open interval described by $m \in (m_e, m_\mu)$ do occur. In the degenerate case $m_e = m_\mu$, the band (24) collapses to the curve $m_H(m_\tau; m_\mu)$ which is the graph corresponding to two generations, equation (21).

In the second case, $W < \mu$, z_1 varies in an infinite interval,

$$\frac{\tau - W}{W - e} < z_1 < \infty,$$

and the Higgs mass is now a decreasing function of z_1. Again, we get two inequalities,

$$3(W - e)\left(\frac{\mu}{W} - 1\right) < H < 3(W - e)\left(\frac{\tau}{W} - 1\right),$$

that reduce to the degenerate case, equation (21), for $\mu = \tau$. Note that these inequalities remain valid for $\mu = W$.

Let us now consider the relation among the gauge couplings g_2 and g_1 in the $\mathbb{H} \oplus \mathbb{C}$ model. For leptons only and any number of generations, we have the *exact* relation [4]

$$g_2 = \sqrt{2}\, g_1, \quad \sin^2\theta = 1/3.$$

If we require the relations among gauge couplings to be fuzzy as well, we must add at least one generation of quarks. Then we get

$$1/5 < \sin^2\theta < 1/3. \tag{25}$$

Note that, if we admit right handed neutrinos, $\sin^2\theta_w = 1/5$ and it can not be made fuzzy by the addition of quarks.

The analysis of the fuzzy mass relations in the presence of quarks is more complicated than the purely leptonic analysis above. To simplify, let us take two generations of leptons, μ and τ, and one generation of quarks, t and b. Then equations (15, 13, 14) read

$$
\begin{aligned}
E &= x + y_2 + y_3, \\
K &= \frac{3}{2}(t^2 + b^2)x + tbx + \frac{3}{2}(\mu^2 y_2 + \tau^2 y_3) \\
&\quad - \frac{1}{2}L^2\left[\frac{1}{x + y_2 + y_3} + \frac{1}{x + (y_2 + y_3)/2}\right], \\
L &= (t + b)x + \mu y_2 + \tau y_3.
\end{aligned} \tag{26}
$$

Even after this simplification, many different cases have to be distinguished whereas in the last example we only had two cases. Let us only treat one case here: if we assume $\mu < \tau < (1 - 1/\sqrt{3})(t + b)$ then the Higgs mass will also be a monotonic function, just as in the three lepton example. As before, equation (16) yields lower and upper bounds for the W mass, $e < W < t + b$. Adapting the analysis of three lepton generations to the present case, we find again that the Higgs mass varies in a finite, open interval, $H_{\min} < H < H_{\max}$ with

$$H_{\min} = 3\left(t + b + \tau - (t + b)\frac{\tau}{W}\right) - W\frac{3(t + b) + W - 4\tau}{t + b + W - 2\tau} - 4t\frac{b}{W}\frac{W - \tau}{t + b - \tau}, \quad (27)$$

$$H_{\max} = 3\left(t + b + \mu - (t + b)\frac{\mu}{W}\right) - W\frac{3(t + b) + W - 4\mu}{t + b + W - 2\mu} - 4t\frac{b}{W}\frac{W - \mu}{t + b - \mu}. \quad (28)$$

Away from the lower bound $m_W^2 - m_b^2$ of m_t^2, the width of the allowed band in the $m_t\, m_H$ plane is again governed by the (light) leptons, in the sense that the band collapses if $m_\tau = m_\mu$.

Let us put back colour and consider the standard model with $N = 3$ generations of leptons and quarks. Note that now in equations (15, 13, 14), all three quark generations are poised with the same positive parameter x, while the three lepton generations are poised independently with the three positive parameters y_1, y_2 and y_3:

$$E = 3x + y_1 + y_2 + y_3, \quad (29)$$

$$\begin{aligned}
K = {} & \frac{3}{2}(u^2 + d^2 + c^2 + s^2 + t^2 + b^2)x + (ud + cs + tb)x \\
& + \frac{3}{2}(e^2 y_1 + \mu^2 y_2 + \tau^2 y_3) \\
& - \frac{1}{2}L^2\left[\frac{1}{3x + y_1 + y_2 + y_3} + \frac{1}{3x + (y_1 + y_2 + y_3)/2}\right], \quad (30)
\end{aligned}$$

$$L = (u + d + c + s + t + b)x + ey_1 + \mu y_2 + \tau y_3. \quad (31)$$

Recall that this difference is due to the quark mixing given by a non-degenerate Cabbibo-Kobayashi-Maskawa matrix. Here non-degenerate means that there are no common mass and weak interactions eigenstates in the quark sector. This reduction of parameters modifies the bounds on the W mass,

$$e < W < (u + d + c + s + t + b)/3$$

otherwise the Cabbibo-Kobayashi-Maskawa matrix drops out of equations (29-31). Note that colour does not affect the W and Higgs masses because of the vector character of strong interactions and because of the homogeneous appearance of the parameters x, y_1, y_2, y_3 in equations (16, 17). In particular, we get lower and upper bounds on the Higgs mass similar to equations (27) and (28) if we restrict ourselves here to the case $\tau < t + b$. Again, putting the lepton masses to zero makes these bounds collapse, the fuzzy mass relation becomes exact,

$$H = \frac{3(u^2 + d^2 + c^2 + s^2 + t^2 + b^2) + 2(ud + cs + tb)}{u + d + c + s + t + b} - 3W\frac{u + d + c + s + t + b + W}{u + d + c + s + t + b + 3W}.$$

The complete analysis will be published elsewhere [15].

Finally let us discuss the relations among gauge couplings in the standard model. The addition of colour changes the picture quite drastically because of the additional element z' in the commutants and because of the strong gauge coupling g_3. Recall the gauge coupling ratios

$$\sin^2 \theta_w = \frac{3x + \operatorname{tr} y}{6x + \frac{3}{2}\operatorname{tr} y + \frac{2}{3}x' + \frac{3}{2}\operatorname{tr} y'},$$

and

$$\left(\frac{g_3}{g_2}\right)^2 = \frac{3x + \operatorname{tr} y}{4x'}.$$

Consequently, the strong gauge coupling is arbitrary. This is natural. However, via the unimodularity condition, it back reacts on the weak mixing angle and

$$\sin^2 \theta_w \quad < \quad \frac{2}{3}\left(1 + \frac{W}{u + d + c + s + t + b} + \frac{1}{9}\left(\frac{g_2}{g_3}\right)^2\right)^{-1} \qquad \text{if } e = \mu = \tau = 0.$$

Numerically, this back reaction is negligible, $(g_2/g_3)^2 = 0.015$. However, for non-zero lepton masses, even for light leptons, the optimal bound of $\sin^2 \theta_w$ reduces to $2/3$ annihilating the mentioned back reaction.

For the natural subclass of scalar products defined with z and z' in $\rho(\text{center } \mathcal{A}) \cap \rho'(\text{center } \mathcal{A}')$ we have

$$y = \frac{x}{3}\,1_3, \quad y' = \frac{x'}{3}\,1_3.$$

Consequently, the fuzziness of the mass relations is lost,

$$W \;=\; (u + d + c + s + t + b)/4 + (e + \mu + \tau)/12,$$

$$H \;=\; \frac{3(u^2 + d^2 + c^2 + s^2 + t^2 + b^2) + 2(ud + cs + tb) + (\tau^2 + \mu^2 + e^2)}{4W} - \frac{15}{7}W.$$

The ratios of gauge couplings reduce to

$$\sin^2 \theta_w = \frac{24}{45 + 13(g_2/g_3)^2} < \frac{8}{15},$$

$$\left(\frac{g_3}{g_2}\right)^2 = \frac{x}{x'}.$$

In conclusion, as a Yang-Mills-Higgs model, the standard model can be accommodated in the very narrow frame of non-commutative geometry under two conditions. The first condition concerns the representation content, fermions must sit exclusively in fundamental or singlet representations and the Higgs scalar sits in *one* weak isospin doublet. The second condition concerns gauge couplings and masses and we find a rich structure. The Higgs mass is determined

by all fermion masses with a conceptual uncertainty of one part per thousand, $m_H = 280 \pm 33\,GeV$ for $m_t = 176 \pm 18\,GeV$. Naturally, we interpret this prediction to hold for pole masses, because the pole masses are gauge invariant. Nevertheless, should the reader be inclined to interpret the relations among masses and gauge couplings at the scale dependent level, then he may do so. Indeed, their inherent 'fuzziness' renders them stable under local renormalization flow and this should be enough as the theory does not contain any super heavy scale [16]. In any case, this rich structure deserves further theoretical and experimental exploration.

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